

AGT, BURGE PAIRS AND MINIMAL MODELS

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ABSTRACT. We consider the AGT correspondence in the context of the conformal field theory $\mathcal{M}^{p,p'} \otimes \mathcal{M}^{\mathcal{H}}$, where $\mathcal{M}^{p,p'}$ is the minimal model based on the Virasoro algebra $\mathcal{V}^{p,p'}$ labeled by two co-prime integers $\{p, p'\}$, $1 < p < p'$, and $\mathcal{M}^{\mathcal{H}}$ is the free boson theory based on the Heisenberg algebra \mathcal{H} . Using Nekrasov's instanton partition functions without modification to compute conformal blocks in $\mathcal{M}^{p,p'} \otimes \mathcal{M}^{\mathcal{H}}$ leads to ill-defined or incorrect expressions.

Let $\mathcal{B}_n^{p,p',\mathcal{H}}$ be a conformal block in $\mathcal{M}^{p,p'} \otimes \mathcal{M}^{\mathcal{H}}$, with n consecutive channels χ_ι , $\iota = 1, \dots, n$, and let χ_ι carry states from $\mathcal{H}_{r_\iota, s_\iota}^{p,p'} \otimes \mathcal{F}$, where $\mathcal{H}_{r_\iota, s_\iota}^{p,p'}$ is an irreducible highest-weight $\mathcal{V}^{p,p'}$ -representation, labeled by two integers $\{r_\iota, s_\iota\}$, $0 < r_\iota < p$, $0 < s_\iota < p'$, and \mathcal{F} is the Fock space of \mathcal{H} .

We show that restricting the states that flow in χ_ι to states labeled by a partition pair $\{Y_1^\iota, Y_2^\iota\}$ such that $Y_{2,R}^\iota - Y_{1,R+s_\iota-1}^\iota \geq 1 - r_\iota$, and $Y_{1,R}^\iota - Y_{2,R+p'-s_\iota-1}^\iota \geq 1 - p + r_\iota$, where $Y_{i,R}^\iota$ is row- R of Y_i^ι , $i \in \{1, 2\}$, we obtain a well-defined expression that we identify with $\mathcal{B}_n^{p,p',\mathcal{H}}$. We check the correctness of this expression for **1**. Any 1-point $\mathcal{B}_1^{p,p',\mathcal{H}}$ on the torus, when the operator insertion is the identity, and **2**. The 6-point $\mathcal{B}_3^{3,4,\mathcal{H}}$ on the sphere that involves six Ising magnetic operators.

1. INTRODUCTION

1.1. AGT in generic models. Consider the two-dimensional conformal field theory $\mathcal{M}^{gen,\mathcal{H}} = \mathcal{M}^{gen} \otimes \mathcal{M}^{\mathcal{H}}$, based on the algebra $\mathcal{V}^{gen,\mathcal{H}} = \mathcal{V}^{gen} \oplus \mathcal{H}$, where \mathcal{M}^{gen} is a generic model with a chiral spectrum that spans infinitely-many infinite-dimensional irreducible highest-weight \mathcal{V}^{gen} -representations¹, $\mathcal{M}^{\mathcal{H}}$ is the conformal field theory of a free boson that takes values in \mathbb{R} , \mathcal{V}^{gen} is the Virasoro algebra of generic central charge c_{gen} ², and \mathcal{H} is the Heisenberg algebra. The Virasoro central charge of $\mathcal{M}^{\mathcal{H}}$ is $c_{\mathcal{H}} = 1$. The AGT correspondence of Alday, Gaiotto and Tachikawa [1] identifies conformal blocks in $\mathcal{M}^{gen,\mathcal{H}}$ [2] with instanton partition functions in four-dimensional $\mathcal{N}=2$ supersymmetric quiver gauge theories [3]. Conjectured in [1], AGT was proven for $c_{gen}=1$ in [4], and for all c_{gen} in [5] for conformal blocks with non-degenerate external primary fields.

1.2. AGT in minimal models. In this note, we consider AGT in the context of $\mathcal{M}^{p,p',\mathcal{H}} = \mathcal{M}^{p,p'} \otimes \mathcal{M}^{\mathcal{H}}$, based on the algebra $\mathcal{V}^{p,p',\mathcal{H}} = \mathcal{V}^{p,p'} \oplus \mathcal{H}$, where $\mathcal{M}^{p,p'}$ is the minimal conformal field theory with a chiral spectrum that spans finitely-many $\mathcal{V}^{p,p'}$ irreps, and $\mathcal{V}^{p,p'}$ is the Virasoro algebra labeled by two co-prime integers $\{p, p'\}$, $0 < p < p'$, of central charge $c_{p,p'}$,

$$(1) \quad c_{p,p'} = 1 - 6 \left(\left(\frac{p}{p'} \right)^{\frac{1}{2}} - \left(\frac{p'}{p} \right)^{\frac{1}{2}} \right)^2$$

Key words and phrases. AGT correspondence. Burge pairs. Minimal Virasoro conformal field theories. Conformal blocks.

¹ Only the chiral sector of a conformal field theory is discussed in this work, and this is implied in the sequel. We abbreviate ‘the AGT correspondence’ to ‘AGT’, and ‘irreducible highest-weight representation’ to ‘irrep’, which in this work is always infinite-dimensional.

² By generic Virasoro central charge c_{gen} we specifically mean $c_{gen} \neq c_{p,p'}$, where $c_{p,p'}$ is the Virasoro central charge of the minimal model $\mathcal{M}^{p,p'}$.

Let $\mathcal{B}_n^{p,p',\mathcal{H}}$, $\mathcal{B}_n^{p,p'}$ and $\mathcal{B}^{\mathcal{H}}$ be conformal blocks with n consecutive channels ³. We wish to compute any $\mathcal{B}_n^{p,p',\mathcal{H}}$ of vertex operators $\mathcal{O}_\iota^{p,p',\mathcal{H}}(z_\iota) = \mathcal{O}_\iota^{p,p'}(z_\iota) \times \mathcal{O}_\iota^{\mathcal{H}}(z_\iota)$, $\iota = 0, \dots, n+2$. Since any $\mathcal{B}_n^{p,p',\mathcal{H}}$ factorizes to $\mathcal{B}_n^{p,p'} \times \mathcal{B}_n^{\mathcal{H}}$ and an explicit expression for the $\mathcal{M}^{\mathcal{H}}$ -factor $\mathcal{B}_n^{\mathcal{H}}$ is known ⁴, computing $\mathcal{B}_n^{p,p',\mathcal{H}}$ is equivalent to computing its $\mathcal{M}^{p,p'}$ -factor $\mathcal{B}_n^{p,p'}$ which is typically what we want.

1.3. Zeros in denominators and deformations. Applying AGT to minimal model conformal blocks without modification leads to ill-defined expressions, as will be explained in detail below. In particular, setting the parameters that appear in Nekrasov's partition functions to minimal model values leads to zeros in the denominators of the summands. Following [6], one can make the summands well-defined using suitable deformations of the parameters. Doing that, one finds whenever a denominator is zero in the limit of removing the deformations, the corresponding numerator is also zero in such a way that and that limit is well-defined. This is in agreement with [7], where arguments were given to the effect that, analytically continuing the conformal blocks in the conformal dimensions of the primary states that flow in each channel, the only singularities are poles and the sum of all residues is zero. This is the approach that was followed, albeit without discussion, in an earlier work on AGT in minimal models [8].

1.4. Zeros in denominators and restrictions. In this note, we follow a different approach from that discussed in subsection 1.3. Our idea is that the zeros in the denominators of Nekrasov's partition functions are due to including null states that should not be included. We avoid this by restricting the summations over Young diagrams that appear in Nekrasov partition functions to avoid these null states. We make the summands well-defined by restricting the partition pairs that label the summed-over states to exclude the summands with poles. To compute $\mathcal{B}_n^{p,p',\mathcal{H}}$, the summations that label the factors in Nekrasov's instanton partition functions must be restricted to avoid ill-defined or incorrect expressions for $\mathcal{B}_n^{p,p',\mathcal{H}}$, and consequently for its $\mathcal{M}^{p,p'}$ -factor $\mathcal{B}_n^{p,p'}$. Our approach allows us to characterise the Young diagrams that label the summands that do contribute to $\mathcal{B}_n^{p,p',\mathcal{H}}$.

1.5. Unrestricted partition pairs. The AGT expression for a linear conformal block $\mathcal{B}_n^{gen,\mathcal{H}}$, that has n consecutive channels χ_ι , $\iota = 1, \dots, n$, is an n -fold sum ⁵,

$$(2) \quad \mathcal{B}_n^{gen,\mathcal{H}} = \sum_{\vec{Y}^1, \dots, \vec{Y}^n} \prod_{\iota=1}^{n+1} q_\iota^{|\vec{Y}^\iota|} Z_{bb}^\iota \left(\vec{a}^{\iota-1}, \vec{Y}^{\iota-1} \mid \mu^\iota \mid \vec{a}^\iota, \vec{Y}^\iota \right),$$

where the summand is a product of $(n+1)$ factors $q_\iota^{|\vec{Y}^\iota|} Z_{bb}^\iota[\vec{a}^{\iota-1}, \vec{Y}^{\iota-1} \mid \mu^\iota \mid \vec{a}^\iota, \vec{Y}^\iota]$, $\iota = 1, \dots, n+1$, that will be defined in section 2. Each factor Z_{bb}^ι is a rational function that depends on two pairs of 'unrestricted' Young diagrams $\{Y_1^{\iota-1}, Y_2^{\iota-1}\}$ and $\{Y_1^\iota, Y_2^\iota\}$. In other words, there are no conditions on these Young diagrams and all possible pairs are allowed. The denominator z_{den}^ι of Z_{bb}^ι is a product of the norms of the states that flow in the preceding channel $\chi^{\iota-1}$ and the subsequent channel χ^ι . Since Z_{bb}^ι is labeled by unrestricted partition pairs, and the sums are over all possible unrestricted pairs, the states that flow in each channel belong to a Verma module of $\mathcal{V}^{gen,\mathcal{H}}$.

Applying AGT without modification to $\mathcal{M}^{p,p',\mathcal{H}}$, one includes zero-norm states in the summation, and thereby includes states in a Verma module rather than in an irrep of $\mathcal{V}^{p,p',\mathcal{H}}$. This leads to summands in the instanton partition function with zero

³ Only linear conformal blocks, as in Figure 3, are considered in this work. Our notation is such that an n -channel conformal block $\mathcal{B}_n^{indices}$, is the expectation value of $(n+3)$ vertex operators $\mathcal{O}_\iota^{same\ indices}(z_\iota)$, $\iota = 0, \dots, (n+2)$, in $\mathcal{M}^{same\ indices}$ on a Riemann surface \mathcal{S} , and $z_\iota \in \mathcal{S}$.

⁴ See, for example, equation (1.9) in [5].

⁵ The partition pairs \vec{Y}^0 and \vec{Y}^{n+1} are trivial, that is they consist of empty partitions, and no summation is performed on them.

denominators. Further, as show, whenever a denominator in a summand vanishes, the corresponding numerator vanishes as well and one ends with ill-defined expressions

1.6. Restricted partition pairs. In this note, we consider $\mathcal{B}_n^{p,p',\mathcal{H}}$ as an instanton partition Z_{Nek} that consists of building block partition functions Z_{bb}^ι that has a numerator z_{num}^ι and a denominator z_{den}^ι , $\iota = 1, \dots, n+1$. Z_{bb}^ι connects two channels $\chi_{\iota-1}$ and χ_ι . The denominator z^ι of Z_{bb}^ι is a product of two factors $[z_{norm}^{\iota-1}]^{1/2}$ and $[z_{norm}^\iota]^{1/2}$ that account for the norms of the states that flow in the channels $\chi_{\iota-1}$ and χ_ι , respectively. We characterise the zeros in these denominators that lead to ill-defined expressions for $\mathcal{B}_n^{p,p',\mathcal{H}}$. If channel χ_ι , $\iota = 1, \dots, n$, carries states that belong to an irreducible highest weight Virasoro representation that flows is $\mathcal{H}_{r_\iota, s_\iota}^{p,p'}$, we attribute these zeros to the flow of null states that do not belong to $\mathcal{H}_{r_\iota, s_\iota}^{p,p'}$, and eliminate these zeros by restricting the partition pairs that appear in Nekrasov's original expressions to partition pairs $\{Y_1, Y_2\}$, that satisfy the conditions

$$(3) \quad \boxed{Y_{2,R}^\iota - Y_{1,R+s_\iota-1}^\iota \geq 1 - r_\iota, \quad Y_{1,R}^\iota - Y_{2,R+p-s_\iota-1}^\iota \geq 1 - p' + r_\iota}$$

where $Y_{i,R}^\iota$ is row-R of Y_i^ι , $i \in \{1, 2\}$.

1.7. Burge pairs. Partition pairs that satisfy conditions (3) were first studied in [9] and appeared more recently in [10, 11]. In this work, we refer to them as *Burge pairs*, and show that when used to restrict AGT to compute $\mathcal{B}_n^{p,p',\mathcal{H}}$, that is when we sum over Burge pairs rather than on all possible partition pairs,

$$(4) \quad \mathcal{B}_n^{p,p',\mathcal{H}} = \sum_{\vec{Y}^1, \dots, \vec{Y}^n} \prod_{\iota=1}^{n+1} q_t^{|\vec{Y}^\iota|} Z_{bb}^\iota \left(\vec{a}^{\iota-1}, \vec{Y}^{\iota-1} \mid \mu^\iota \mid \vec{a}^\iota, \vec{Y}^\iota \right)$$

where \sum' indicates that the sum is restricted to partition pairs that satisfy the Burge conditions (3), we obtain well-defined expressions. We check these expressions in two cases **1.** Any 1-point $\mathcal{B}_1^{p,p',\mathcal{H}}$ on the torus, when the operator insertion is the identity, and **2.** The 6-point $\mathcal{B}_3^{3,4,\mathcal{H}}$, when all operator insertions involve Ising magnetic operators. We also give arguments why we expect this identification to be correct.

1.8. Outline of contents. In section **2**, we recall basic facts related to Nekrasov's instanton partition functions. In **3**, we recall the AGT parametrisation of $\mathcal{M}^{gen,\mathcal{H}}$, the choice of parameters that allows us to obtain $\mathcal{M}^{p,p',\mathcal{H}}$, then show how the unrestricted instanton partition functions give the wrong answer in the case of $\mathcal{B}_1^{p,p',\mathcal{H}}$ on the torus. In **4**, we use the requirement that the summands remain well-defined to characterise the partition pairs that label them. We identify these partition functions with $\mathcal{B}_n^{p,p',\mathcal{H}}$. In **5**, we study the vanishing of the numerator, and show that whenever the denominator of a summand vanishes, then the numerator also vanishes. In **6**, we check the correctness of our expressions in the two cases listed above. In **7**, we use results from [5, 10, 11], Proposition **4.1** in section **4**, and Conjecture **7.1** in section **7**, to explain why the restriction to Burge pairs produces conformal blocks in $\mathcal{M}^{p,p',\mathcal{H}}$. Because we use Conjecture 7.1, this explanation is not a proof. In **8**, we extend of our results to conformal blocks in $\mathcal{M}^{gen,\mathcal{H}}$, with degenerate intermediate Virasoro representations, and in **9**, we collect a number of remarks that include **1.** a conjectural generalization to the W_N conformal blocks, and **2.** a geometric interpretation of the summation over Burge pairs as a summation over *isolated* torus fixed points on the instanton moduli space.

2. THE INSTANTON PARTITION FUNCTION

2.1. Partitions. A partition π of an integer $|\pi|$ is a set of non-negative integers $\{\pi_1, \pi_2, \dots, \pi_p\}$, where p is the number of parts, $\pi_i \geq \pi_{i+1}$, and $\sum_{i=1}^p \pi_i = |\pi|$. π is represented as a Young diagram Y , which is a set of p rows $\{Y_1, Y_2, \dots, Y_p\}$, such that row- i has $Y_i = \pi_i$ cells⁶, $Y_i \geq Y_{i+1}$, and $|Y| = \sum_i Y_i = |\pi|$. We use Y_i^\top for the transpose of Y_i , and define $Y_{i,R}^+ = Y_{i,R} + 1$.

We use \square for a cell in a Young diagram Y , which is a square in the south-east quadrant of the plane, with coordinates $\{R, C\}$, such that R is the row-number, counted from top to bottom, and C is the column number, counted from left to right. We define $A_{\square, Y_i}^+ = A_{\square, Y_i} + 1$, where A_{\square, Y_i} is the arm of \square in Y_i , that is, the number of cells in the same row as, but to the right of \square in Y_i , and L_{\square, W_j} to be the leg of \square with respect to its position in W_j , that is the number of cells in the same column as, but below \square in Y_i .

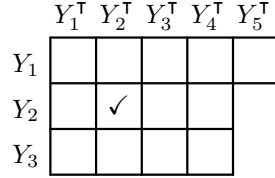


Figure 1. The Young diagram Y of $5+4+4$. The rows are numbered from top to bottom. The rows of the transpose Y^\top that represents $3+3+3+3+1$, which are the columns of Y , are numbered from left to right. From the viewpoint of Y , the marked cell \checkmark has $A_{\checkmark, Y} = 2$, $A_{\checkmark, Y}^+ = 3$, and $L_{\checkmark, Y} = 1$. From the viewpoint of Y^\top , \checkmark has $A_{\checkmark, Y^\top} = 1$, $A_{\checkmark, Y^\top}^+ = 2$, and $L_{\checkmark, Y^\top} = 2$.

2.2. Partition pairs. The AGT representation of $\mathcal{B}^{p, p', \mathcal{H}}$ involves a multi-sum over internal states labeled by $n+2$ partition pairs \vec{Y}^ι , $\iota = 0, 1, \dots, n, n+1$, where \vec{Y}^ι is a pair of Young diagrams, $\{Y_1^\iota, Y_2^\iota\}$, and $|\vec{Y}^\iota| = |Y_1^\iota| + |Y_2^\iota|$ is the total number of cells in \vec{Y}^ι . The pairs $\{Y_1^\iota, Y_2^\iota\}$, $\iota \in \{1, \dots, n\}$, are non-empty Young diagrams, while $\{Y_1^\iota, Y_2^\iota\}$, $\iota \in \{0, n+1\}$ are empty⁷, $\vec{Y}^{(0)} = \vec{Y}^{(n+1)} = \vec{\emptyset}$, where $\vec{\emptyset}$ is a pair of empty Young diagrams.

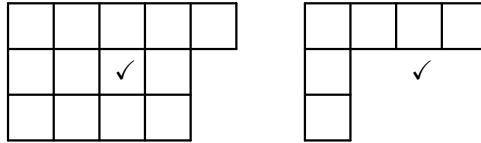


Figure 2. A partition pair $\{Y_1, Y_2\}$. Y_1 is on the left, Y_2 is on the right. The cell \checkmark has coordinates $(2, 3)$, $\checkmark \in Y_1$, but $\checkmark \notin Y_2$. It has $A_{\checkmark, Y_1} = 1$, $A_{\checkmark, Y_1}^+ = 2$, $L_{\checkmark, Y_1} = 1$, as well as $A_{\checkmark, Y_2} = -2$, $A_{\checkmark, Y_2}^+ = -1$, and $L_{\checkmark, Y_2} = -1$.

⁶ We use Y_i for row- i as well as for the number of cells in that row.

⁷ We work in terms of $n+2$ linearly-ordered partition pairs. Since we consider conformal blocks of primary fields, the initial and final pairs are always empty, but we prefer to work in terms of $n+2$ rather than n non-empty pairs to make the notation in the sequel more uniform.

2.3. A decomposition of the instanton partition function. Consider the four-dimensional $\mathcal{N} = 2$ supersymmetric linear quiver gauge theory with a gauge group $\prod_{\iota=1}^{n+1} U(2)_{\iota}$, that is $(n+1)$ copies of $U(2)$ [3]. The instanton partition function of this theory can be written in terms of ‘building block’ partition functions Z_{bb}^{ι} , $\iota = 1, \dots, n+1$, as follows

$$(5) \quad Z_{Nek} = \sum_{\vec{Y}^1, \dots, \vec{Y}^n} \prod_{\iota=1}^{n+1} q_{\iota}^{|\vec{Y}^{\iota}|} Z_{bb}^{\iota} \left(\vec{a}^{\iota-1}, \vec{Y}^{\iota-1} \mid \mu^{\iota} \mid \vec{a}^{\iota}, \vec{Y}^{\iota} \right),$$

where q_{ι} is an indeterminate. In gauge theory, $q_{\iota} = e^{2\pi i \tau_{\iota}}$, where τ_{ι} is the complexified coupling constant of $U(2)_{\iota}$. In conformal field theory, it is a rational function of the positions z_{ι} , $\iota = 0, 1, \dots, n+2$, of the vertex operators \mathcal{O}_{ι} , whose expectation value is the conformal block, on the Riemann surface \mathcal{S} that the conformal field theory is defined on. Z_{bb}^{ι} is defined in subsection 2.4.

The decomposition of the instanton partition function in (5) follows that in [12] and mirrors the decomposition of conformal blocks on a sphere, represented as a *comb diagram* in Figure 3.

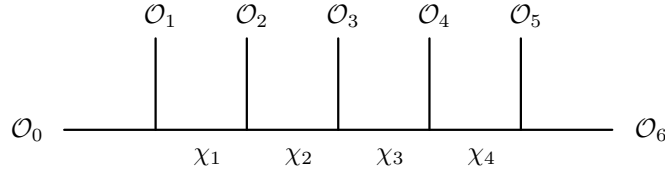


Figure 3. The comb diagram of a 4-channel conformal block that corresponds to a linear quiver. It consists of an initial state that corresponds to a vertex operator \mathcal{O}_0 on the left, five vertex operator insertions $\mathcal{O}_1, \dots, \mathcal{O}_5$, and a final state that corresponds to a vertex operator \mathcal{O}_6 on the right. \mathcal{O}_{ι} is placed at z_{ι} , where z_0, z_{n+1} and z_{n+2} are set 0, 1 and ∞ , respectively. In this example, $n = 4$.

2.4. The building block of the instanton partition function. Z_{bb} is

$$(6) \quad Z_{bb} \left(\vec{a}, \vec{Y} \mid \mu \mid \vec{b}, \vec{W} \right) = \frac{z_{num} \left(\vec{a}, \vec{Y} \mid \mu \mid \vec{b}, \vec{W} \right)}{z_{den} \left(\vec{a}, \vec{Y} \mid \vec{b}, \vec{W} \right)},$$

The parameters that appear in Z_{bb} are as follows.

2.4.1. The 2-component vector $\vec{a}^{\iota} = \{a^{\iota}, -a^{\iota}\}$. In gauge theory, a^{ι} is the expectation value of the vector multiplet in the adjoint representation of the gauge group $U(2)_{\iota}$. In conformal field theory, a^{ι} is the charge of the highest weight of the Virasoro irrep that flows in channel χ_{ι} in the conformal block under consideration.

2.4.2. The partition pairs \vec{Y} and \vec{W} . In gauge theory, each partition pair $\vec{Y}^{\iota} = \{Y_1^{\iota}, Y_2^{\iota}\}$ labels the fixed localization points in the instanton moduli space of $U(2)_{\iota}$. In conformal field theory, they label the states that flow in channel χ_{ι} in the corresponding conformal block. In (6), \vec{Y} and \vec{W} are attached to the line segments on the left and the right of a given vertex, respectively.

2.4.3. The scalar μ^{ι} . In gauge theory, μ^{ι} is the mass parameter of the bi-fundamental matter field that interpolates the gauge groups $U(2)_{\iota}$ and $U(2)_{\iota+1}$. In conformal field theory, μ^{ι} is the charge of the vertex operator that connects channels χ_{ι} and $\chi_{\iota+1}$. In the following, we study the structure of the right hand side of (6).

2.4.4. The denominator.

$$(7) \quad z_{den} \left(\vec{a}, \vec{Y} \mid \vec{b}, \vec{W} \right) = \left(z_{norm} \left(\vec{a}, \vec{Y} \right) z_{norm} \left(\vec{b}, \vec{W} \right) \right)^{\frac{1}{2}},$$

where

$$(8) \quad z_{norm} \left(\vec{a}, \vec{Y} \right) = z_{num} \left(\vec{a}, \vec{Y} \mid 0 \mid \vec{a}, \vec{Y} \right)$$

In gauge theory, z_{norm} is a normalization factor related to the contribution of the vector multiplets that the bi-fundamental couples to. In conformal field theory, it accounts for the norms of the states that propagate into and out of the vertex operator insertion in Z_{bb} .

2.4.5. The numerator.

$$(9) \quad z_{num} \left(\vec{a}, \vec{Y} \mid \mu \mid \vec{b}, \vec{W} \right) = \prod_{i,j=1}^2 \prod_{\square \in Y_i} \left(E[a_i - b_j, Y_i, W_j, \square] - \mu \right) \prod_{\blacksquare \in W_j} \left(\epsilon_1 + \epsilon_2 - E[b_j - a_i, W_j, Y_i, \blacksquare] - \mu \right),$$

where the elementary function $E[x, Y_i, W_j, \square]$ is defined as

$$(10) \quad E[x, Y_i, W_j, \square] = x + A_{\square, Y_i}^+ \epsilon_2 - L_{\square, W_j} \epsilon_1,$$

x is an indeterminate, and $\{\epsilon_1, \epsilon_2\}$ are Nekrasov's deformation parameters, which are generally complex. In gauge theory, z_{num} is the contribution of a bi-fundamental multiplet in $U(2)_\iota$ and $U(2)_{\iota+1}$. In conformal field theory, it is the contribution of the vertex operator insertion that inputs a charge μ into the conformal block into Z_{bb} .

2.4.6. Remark. One can think of z_{num} as the basic object in $U(2)$ AGT theory and in this paper, and all other objects can be written in terms of special cases of it.

2.4.7. Normalisation. Consider the special case where the Virasoro part of the vertex operator in Z_{bb} is the identity, that is $\{r, s\} = \{1, 1\}$, and consequently $\mu = 0$ ⁸. Z_{bb} is defined combinatorially and does not necessarily vanish when the fusion rules are not satisfied. To ensure that the fusion rules are satisfied, we set $\vec{a} = \vec{b}$.

Setting $\mu = 0$ and $\vec{a} = \vec{b}$ ensures that the Virasoro part of the vertex operator insertion is the identity operator. However, one can show that, in this case, the Heisenberg part of the vertex operator is an exponential of the creation part of the free boson field [5], which in general contributes to a difference between \vec{Y} and \vec{W} , and therefore we do not necessarily have $\vec{Y} = \vec{W}$. Setting $\vec{Y} = \vec{W}$, we pick up the contribution of the trivial part of the exponential, that is the identity, and Z_{bb} reduces to

$$(11) \quad Z_{bb} \left(\vec{a}, \vec{Y} \mid 0 \mid \vec{a}, \vec{Y} \right) = \frac{z_{num} \left(\vec{a}, \vec{Y} \mid 0 \mid \vec{a}, \vec{Y} \right)}{z_{den} \left(\vec{a}, \vec{Y} \mid \vec{a}, \vec{Y} \right)} = 1$$

Equation (11) is relevant to computing 1-point conformal blocks of the identity operator on the torus in subsections **3.4** and **4.15**.

⁸ See subsection **5.2**.

3. UNRESTRICTED INSTANTON PARTITION FUNCTIONS FOR $\mathcal{M}^{p,p',\mathcal{H}}$

3.1. AGT parameterisation. Generic models. A generic model is a conformal field theory characterised by a central charge c_{gen} that we parametrise as

$$(12) \quad c_{gen} = 1 + 6 \left(b_{gen} + \frac{1}{b_{gen}} \right)^2, \quad b_{gen} = \left(\frac{\epsilon_2}{\epsilon_1} \right)^{\frac{1}{2}},$$

In the Coulomb gas approach to computing conformal blocks in generic models, the screening charges $\{\beta_+, \beta_-\}$, and the background charge, $-2\beta_0$, satisfy⁹

$$(13) \quad \beta_+ = b_{gen}, \quad \beta_- = \frac{1}{b_{gen}}, \quad 2\beta_0 = \beta_+ + \beta_-$$

3.2. AGT parameterisation. Minimal models. A minimal model $\mathcal{M}^{p,p'}$, based on a Virasoro algebra $\mathcal{V}^{p,p'}$, characterised by a central charge $c_{p,p'} < 1$, that we parameterise as

$$(14) \quad c_{p,p'} = 1 - 6 \left(a_{p,p'} - \frac{1}{a_{p,p'}} \right)^2, \quad a_{p,p'} = \left(\frac{p'}{p} \right)^{\frac{1}{2}},$$

where $\{p, p'\}$ are the minimal model parameters, which are co-prime integers and satisfy $0 < p < p'$, in our conventions. In the Coulomb gas approach to computing conformal blocks in minimal models with $c < 1$ [13, 14], the screening charges $\{\alpha_+, \alpha_-\}$, and the background charge, $-2\alpha_0$, satisfy

$$(15) \quad \alpha_+ = a_{p,p'}, \quad \alpha_- = -\frac{1}{a_{p,p'}}, \quad 2\alpha_0 = \alpha_+ + \alpha_-$$

The AGT parameterisation of $\mathcal{M}^{p,p',\mathcal{H}}$ is obtained by choosing

$$(16) \quad \epsilon_1 < 0 < \epsilon_2, \quad \epsilon_1 = \alpha_-, \quad \epsilon_2 = \alpha_+$$

so that $\alpha_- < 0 < \alpha_+$. Since we focus on $\mathcal{M}^{p,p',\mathcal{H}}$, we work in terms of $\{\alpha_-, \alpha_+\}$ instead of $\{\epsilon_1, \epsilon_2\}$, and write the elementary function $E[x, Y_i, W_j, \square]$ as

$$(17) \quad \boxed{E[x, Y_i, W_j, \square] = x + A_{\square,i}^+ \alpha_+ - L_{\square,j} \alpha_-}$$

3.3. Charge content. We need two distinct objects that, in Coulomb gas terms, are expressed in terms of the screening charges $\{\alpha_+, \alpha_-\}$. **1.** The charge $\mu_{r,s}$ of the vertex operator \mathcal{O}_μ that intertwines two irrep's $\mathcal{H}_{r_1,s_1}^{p,p'}$ and $\mathcal{H}_{r_2,s_2}^{p,p'}$, and **2.** The highest weight $|a_{r,s}\rangle$ of an irrep $\mathcal{H}_{r,s}^{p,p'}$. Following [1, 5], we use $\{r, s\}$ as indices for the charge $\mu_{r,s}$ of the vertex operator $\mathcal{O}_{\mu_{r,s}}$, and $\{r, s\}$ as indices for the charge $a_{r,s}$ of the highest weight $|a_{r,s}\rangle$. These charges are parameterised in terms of α_+ and α_- as follows

$$(18) \quad \mu_{r,s} = - \left(\frac{r-1}{2} \right) \alpha_+ - \left(\frac{s-1}{2} \right) \alpha_-,$$

$$a_{r,s} = -\frac{r}{2} \alpha_+ - \frac{s}{2} \alpha_-, \quad 1 \leq r \leq p-1, 1 \leq s \leq p'-1$$

Note that the same numerical values of $\{r, s\}$ indicate different charge contents in $\mu_{r,s}$ and in $a_{r,s}$. In particular,

⁹ We use $\beta_+, \beta_-, -2\beta_0$ for generic model charges and reserve α_+, α_- and $-2\alpha_0$ for the corresponding minimal model charges. We use b_{gen} and $a_{p,p'}$ for the parameters used to describe the generic and minimal models central charges respectively, since a and b are used for other purposes in the sequel.

$$(19) \quad 2\mu_{r,s} = 2a_{r,s} + (\alpha_+ + \alpha_-)$$

3.4. Unrestricted instanton partition functions give incorrect 1-point functions on the torus. Consider a conformal block in a 1-point function in $\mathcal{M}^{p,p',\mathcal{H}}$ on a torus, Figure 4.

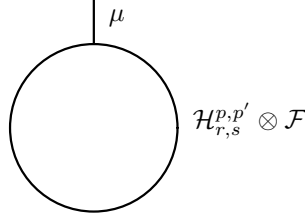


Figure 4. A graph of a conformal block of a 1-point function on a torus. The vertex operator insertion carries a charge μ . The chiral states that flow in the channel belong to $\mathcal{H}_{r,s}^{p,p'} \otimes \mathcal{F}$. For this 1-point function to be finite, the fusion rules must be satisfied.

Following [1], this is given by the instanton partition function of the $\mathcal{N} = 2^* U(2)$ theory,

$$(20) \quad Z_{Nek}^{\mathcal{N}=2^*, U(2)}(\vec{a}, \vec{\mu}) = \sum_{\vec{Y}} q^{|\vec{Y}|} \frac{z_{num}(\vec{a}, \vec{Y} \mid \mu \mid \vec{a}, \vec{Y})}{z_{den}(\vec{a}, \vec{Y} \mid \vec{a}, \vec{Y})},$$

where μ is determined by the operator insertion, and \vec{a} is determined by the states of the $\mathcal{H}_{r,s}^{p,p'}$ that flow in the torus and determine the conformal block. When the inserted operator is the identity, that is $\{r, s\}$, then¹⁰ $\mu = 0$, and if \vec{Y} is an unrestricted partition pair as in the original AGT prescription, then

$$(21) \quad Z_{Nek}^{\mathcal{N}=2^*, U(2)}(\vec{a}, \vec{0}) = \sum_{\vec{Y}} q^{|\vec{Y}|} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^2}$$

while the correct result is

$$(22) \quad Z_{Nek}^{\mathcal{N}=2^*, U(2)}(\vec{a}, \vec{0}) = \frac{\chi_{r,s}^{p,p'}}{\prod_{n=1}^{\infty} (1 - q^n)},$$

where $\chi_{r,s}^{p,p'}$ is the character of the irrep $\mathcal{H}_{r,s}^{p,p'}$ that flows in the torus, and $\vec{a} = \{a, -a\}$, where $2a = -r\alpha_+ - s\alpha_-$,

$$(23) \quad \chi_{r,s}^{p,p'} = \frac{\sum_{k=-\infty}^{\infty} \left(q^{k^2 pp' + k(p'r - ps)} - q^{(kp+r)(kp'+s)} \right)}{\prod_{n=1}^{\infty} (1 - q^n)}$$

This simple example makes it clear that applying the prescription of [1] to $\mathcal{M}^{p,p',\mathcal{H}}$ without modification, leads to incorrect answers. In the following section, we find that it leads to zeros in the denominators of the summands.

¹⁰ See subsection 5.2.

4. RESTRICTED INSTANTON PARTITION FUNCTIONS FOR $\mathcal{M}^{p,p',\mathcal{H}}$.
THE DENOMINATOR

Consider the denominator z_{den} of Z_{bb} in (6). To look for zeros in z_{den} , it is sufficient to look for zeros in $z_{norm}[\vec{a}, \vec{Y}]$ in (8). Consider $\mathcal{B}_n^{p,p',\mathcal{H}}$ and focus on a channel that carries states that belong to $\mathcal{H}_{r,s}^{p,p'}$.

Proposition 4.1. $z_{norm}[\vec{a}, \vec{Y}] \neq 0$, if and only if

$$(24) \quad Y_{2,R} - Y_{1,R+s-1} \geq 1 - r, \quad Y_{1,R} - Y_{2,R+[p'-s]-1} \geq 1 - (p - r),$$

where $Y_{i,R}$ is row-R in Y_i , $i \in \{1, 2\}$.

The proof of Proposition 4.1 is based on checking the products that appear in $z_{norm}[\vec{a}, \vec{Y}]$ for zeros.

4.1. More notation. We set $a_1 = -a_2 = a$, and $a_1 - a_2 = 2a$. If a channel χ_ℓ carries states from $\mathcal{H}_{r_\ell, s_\ell}^{p,p'} \otimes \mathcal{F}$, then the label a_ℓ of the corresponding highest weight is

$$(25) \quad a_{r_\ell, s_\ell} = -r_\ell \alpha_+ - s_\ell \alpha_-$$

4.2. Two zero-conditions. In the sequel, we find that an instanton partition function has a zero when an equation of type

$$(26) \quad C_+ \alpha_+ + C_- \alpha_- = 0,$$

where $\alpha_- < 0 < \alpha_+$, is satisfied. Equivalently, an instanton partition function has a zero when the two conditions

$$(27) \quad C_+ = c p, \quad C_- = c p',$$

are satisfied, where c is some constant that needs to be determined. Given two conditions, such as (27), we need, for the purposes of comparing with known results, to re-write them as one condition.

4.3. From two zero-conditions to one zero-condition. Consider the two conditions

$$(28) \quad \boxed{A_{\square,1} = A' \geq 0, \quad -L_{\square,2} = L' \geq 0}$$

which are satisfied if $\square \in Y_1$, and $\square \notin Y_2$ ¹¹. If \square is in row-R and column-C in Y_1 , then the first condition in (28) implies that there is a cell $\boxplus \in Y_1$, to the right of \square , with coordinates $\{R, C + A'\}$, that lies on a vertical boundary. In other words, **1.** there are no cells to the right of \boxplus , and **2.** there may or may not be cells below \boxplus . This means that column-(C + A') in Y_1 , or equivalently, row-(C + A') in Y_1^\top , has length *at least* R,¹²

$$(29) \quad Y_{1,C+A'}^\top \geq R$$

From the definition of $L_{\square,2}$, we write the second condition in (28) as $-L_{\square,2} = L' = R - Y_{2,C}^\top$, that is $R = L' + Y_{2,C}^\top$, and using (29), we obtain $Y_{1,C+A'}^\top \geq L' + Y_{2,C}^\top$, which we choose to write as

$$(30) \quad Y_{1,C+A'}^\top - Y_{2,C}^\top \geq L',$$

¹¹ We chose the labels of the Young diagrams to be concrete. The same arguments apply under $Y_1 \Leftrightarrow Y_2$.

¹² All equations and inequalities in the sequel involve rows of Young diagrams, and never columns. For example, $Y_{1,C+A'}^\top$ is row-(C + A') in diagram Y_1^\top , which is the transpose of Y_1 . The subscript C is there only because the corresponding row is a column in a diagram that we started our arguments with.

which is one condition that is equivalent to the two conditions in (28).

4.4. One non-zero condition. Consider a function $z[Y_1, Y_2]$, of a pair of Young diagrams $\{Y_1, Y_2\}$, such that $z[Y_1, Y_2] = 0$, *if and only if* (30) is satisfied. This implies that $z[Y_1, Y_2] \neq 0$, *if and only if* $\{Y_1, Y_2\}$ satisfies the complementary condition

$$(31) \quad Y_{1, \mathbf{c}+\mathbf{A}'}^\top - Y_{2, \mathbf{c}}^\top < L',$$

which we choose to write as

$$(32) \quad \boxed{Y_{2, \mathbf{c}}^\top - Y_{1, \mathbf{c}+\mathbf{A}'}^\top \geq 1 - L'}$$

4.4.1. Remark. Since we use equations such as (28) and (32) frequently in the sequel, refer to the former as ‘zero-conditions’, and to the latter as ‘non-zero-conditions’.

4.5. Products that appear in the denominator. Two types of products appear in z_{norm} , **1.** products in the form $\prod_{\square \in Y_i} E[a_i - a_j, Y_i, Y_j, \square]$ that we refer to as $\{Y_i, Y_j\}_{den}$, and **2.** products in the form $\prod_{\square \in Y_i} [\alpha_+ + \alpha_- - E[a_i - a_j, Y_i, Y_j, \square]]$ that we refer to as $\{Y_i, Y_j\}'_{den}$.

4.6. In search of zeros. In the following subsections, **1.** we consider the products that appear in z_{den} , one at a time, **2.** we search for possible zeros, as in subsection 4.2, **3.** we find the conditions that we need to impose on the pair $\{Y_1, Y_2\}$ in order to avoid the zeros, and **4.** when there is more than one set of conditions to avoid the zeros, we choose the stronger set. That is, the set that ensures that all zeros are eliminated. We use the fact that r , s , $p - a$ and $p' - s$ are non-zero positive integers.

4.7. $\{\mathbf{Y}_1, \mathbf{Y}_1\}_{den}$. This product does not vanish, since this requires that there is a factor that satisfies

$$(33) \quad E[0, Y_1, Y_1, \square] = A_{\square, 1}^+ \alpha_+ - L_{\square, 1} \alpha_- = 0,$$

which is not possible since $\square \in Y_1$ and $\alpha_- < 0$.

4.8. $\{\mathbf{Y}_1, \mathbf{Y}_1\}'$, $\{\mathbf{Y}_2, \mathbf{Y}_2\}_{den}$ and $\{\mathbf{Y}_2, \mathbf{Y}_2\}'_{den}$. These products do not vanish for the same reason that $\{Y_1, Y_1\}_{den}$ in paragraph 4.7 does not vanish.

4.9. $\{\mathbf{Y}_1, \mathbf{Y}_2\}_{den}$. This product vanishes if any factor satisfies

$$(34) \quad E[-r \alpha_+ - s \alpha_-, Y_1, Y_2, \square] = \left(-r + A_{\square, 1}^+\right) \alpha_+ + \left(-s - L_{\square, 2}\right) \alpha_- = 0,$$

which lead to the conditions

$$(35) \quad A_{\square, 1} = r - 1 + c p \quad -L_{\square, 2} = s + c p'$$

Since $A_{\square, i}$, $L_{\square, i}$, $i \in \{1, 2\}$, r , and s are non-zero positive integers, and p and p' are positive co-primes, c must be a non-negative integer, and the conditions in (35) are possible for $c = \{0, 1, \dots\}$, $\square \in Y_1$ and $\square \notin Y_2$ ¹³.

4.9.1. From two zero-conditions to one non-zero-condition. Following paragraphs 4.3 and 4.4, the two zero-conditions in (35) can be translated to one non-zero-condition,

$$(36) \quad Y_{1, \mathbf{c}}^\top - Y_{2, \mathbf{c}+\mathbf{r}-1+\mathbf{c}\mathbf{p}}^\top \geq 1 - s - c p'$$

¹³ Note that from conditions (35), if the Young diagram Y such that $\square \in Y$, which in this case is Y_1 , is sufficiently large compared to the Young diagram W such that $\square \notin W$, which in this case is Y_2 , then the product under discussion will have more than one zero. This will be the case in the rest of the factors discussed in this section as well.

4.9.2. *The stronger condition.* Equation (36) is the statement that to eliminate the zeros, we want $Y_{2,c}^\top - Y_{1,c+r-1+c}^\top \geq 1 - s - cp'$, where $c = \{0, 1, \dots\}$. Since the row-lengths of a partition are by definition weakly decreasing, and $c = \{0, 1, \dots\}$, this is the case if $Y_{2,c}^\top - Y_{1,c+p-r-1}^\top \geq 1 - s - cp'$, which is the case if $Y_{2,c}^\top - Y_{1,c+p-r-1}^\top \geq 1 - s$. Thus, we should set $c = 0$, and obtain

$$(37) \quad Y_{2,c}^\top - Y_{1,c+p-r-1}^\top \geq 1 - s$$

4.10. **The 1st Burge condition.** Condition (37) says that if we delete the top $(r-1)$ rows and the left-most $(s-1)$ columns of Y_1^\top , to obtain a reduced Young diagram that we refer to as $Y_{red,1}^\top$, then

$$(38) \quad Y_{2,c}^\top - Y_{red,1,c}^\top \geq 0$$

which implies

$$(39) \quad Y_{2,R} - Y_{red,1,R} \geq 0$$

where Y_R is row- R of Y . Transposing the $(r-1)$ rows and $(s-1)$ columns that we removed earlier from Y_1^\top in order to obtain $Y_{red,1}^\top$, we obtain $(r-1)$ columns and $(s-1)$ rows that we can add to the top and to the left of $Y_{1,red}$, respectively, to obtain

$$(40) \quad \boxed{Y_{2,R} - Y_{1,R+s-1} \geq 1 - r}$$

which is the 1st Burge condition in (3).

4.11. $\{\mathbf{Y}_2, \mathbf{Y}_1\}_{den}$. This product vanishes if any factor satisfies

$$(41) \quad E[r\alpha_+ + s\alpha_-, Y_2, Y_1, \square] = \left(r + A_{\square,2}^+\right) \alpha_+ + \left(s - L_{\square,1}\right) \alpha_- = 0,$$

which leads to the conditions

$$(42) \quad A_{\square,2} = -1 - r + cp, \quad -L_{\square,1} = -s + cp',$$

which are possible for $c = \{1, 2, \dots\}$, $\square \in Y_2$ and $\square \notin Y_1$.

4.11.1. *From two zero-conditions to one non-zero-condition.* Following paragraphs 4.3 and 4.4, the two zero-conditions in (42) can be translated to one non-zero-condition,

$$(43) \quad Y_{1,c}^\top - Y_{2,c-1-r+cp}^\top \geq 1 + s - cp'$$

4.11.2. *The stronger condition.* Equation (43) is the statement that to eliminate the zeros, we want $Y_{2,c}^\top - Y_{1,c-1-r+cp}^\top \geq 1 + s - cp'$, where $c = \{1, 2, \dots\}$. Since the row-lengths of a partition are by definition weakly decreasing, and $c = \{1, 2, \dots\}$, this is the case if $Y_{2,c}^\top - Y_{1,c+p-r-1}^\top \geq 1 - s - cp'$, which is the case if $Y_{2,c}^\top - Y_{1,c+p-r-1}^\top \geq 1 - p + s$. Thus, we should set $c = 1$, to obtain

$$(44) \quad Y_{1,c}^\top - Y_{2,c+[p-r]-1}^\top \geq 1 - (p' - s)$$

4.12. **The 2nd Burge condition.** Following the argument in subsection 4.10, we can write condition (44) as

$$(45) \quad \boxed{Y_{1,R} - Y_{2,R+[p'-s]-1} \geq 1 - (p - r)}$$

which is the 2nd Burge condition in (3).

4.13. $\{\mathbf{Y}_1, \mathbf{Y}_2\}'_{den}$. This product vanishes if any factor satisfies

$$(46) \quad -\alpha_+ - \alpha_- + E[-r\alpha_+ - s\alpha_-, Y_1, Y_2, \square] = \\ \left(-r + A_{\square,1}\right) \alpha_+ + \left(-s - L_{\square,2}^+\right) \alpha_- = 0,$$

which leads to the conditions

$$(47) \quad A_{\square,1} = r + cp, \quad -L_{\square,2} = 1 + s + cp',$$

which, using the same arguments as in subsections 4.9 and 4.11, are possible for $c = \{0, 1, \dots\}$, $\square \in Y_1$, $\square \notin Y_2$, and we should choose $c = 0$ to obtain

$$(48) \quad Y_{2,c}^\top - Y_{1,c+r}^\top \geq -s$$

Comparing condition (37) with condition (48), we see that the former is stronger than the latter, for the same reasons as in paragraph 4.9.2. Thus this case does not offer new conditions on the partition pair.

4.14. $\{\mathbf{Y}_2, \mathbf{Y}_1\}'_{den}$. This product vanishes if any factor satisfies

$$(49) \quad -\alpha_+ - \alpha_- + E[r\alpha_+ + s\alpha_-, Y_2, Y_1, \square] = \\ \left(r + A_{\square,2}\right) \alpha_+ + \left(s - L_{\square,1}^+\right) \alpha_- = 0,$$

which leads to the conditions

$$(50) \quad A_{\square,2} = -r + cp, \quad -L_{\square,1} = 1 - s + cp',$$

which, using the same arguments as in subsections 4.9 and 4.11, are possible for $c = \{1, 2, \dots\}$, $\square \in Y_2$ and $\square \notin Y_1$, and we should choose $c = 1$ to obtain

$$(51) \quad Y_{2,c}^\top - Y_{1,c+[p-r]}^\top \geq -(p' - s)$$

Comparing condition (44) with condition (51), we see that the former is stronger than the latter, for the same reasons as in paragraph 4.11.2. Thus this case does not offer new conditions on the partition pair.

4.15. Restricted instanton partition functions give the correct 1-point function on the torus. From the discussion in paragraphs 4.7–4.14, we conclude that z_{den} has no zeros if the conditions in (37) and (44) are satisfied. As mentioned in section 1, these conditions on partition pairs are known. They were introduced and studied in [9], and were further studied and called *Burge pairs* in [10]. A full and explicit derivation of the fact that the generating function of the Burge pairs, that satisfy conditions (37) and (44), is the q -series in (22), we refer the reader to Appendix A of [10].

4.15.1. Remark. The conditions obtained in this note were written differently in [9, 10] for three reasons. **1.** These papers used the notation $\{a, b, \alpha, \beta\}$, which in terms of the variables $\{r, s, p, p'\}$ used in this work are $a = r, b = p - r, \alpha = s$, and $\beta = p' - s$. **2.** The partition rows were labeled such that $Y_i \geq Y_{i-1}$, while in this note, we assume the opposite (and more conventional) labeling, and **3.** The conventions in [9, 10] are such that the conditions were expressed in terms of the Young diagrams that the presentation naturally started with, while in this work, we wished to follow the conventions of [1, 3, 5], so we ended up expressing the conditions on the partition pair $\{Y_1, Y_2\}$ as conditions on $\{Y_1^\top, Y_2^\top\}$.

5. RESTRICTED INSTANTON PARTITION FUNCTIONS FOR $\mathcal{M}^{p,p',\mathcal{H}}$.
THE NUMERATOR

5.1. Products that appear in the numerator. Two types of products appear in z_{num} , **1.** products in the form $\prod_{\square \in Y_i} [E[x, Y_i, W_j, \square] - \mu]$ that we refer to as $\{Y_i, W_j\}_{num}$, and **2.** products in the form $\prod_{\square \in W_j} [\alpha_+ + \alpha_- - E[-x, W_j, Y_i, \square] - \mu]$ that we refer to as $\{W_j, Y_i\}'_{num}$. We need to examine the conditions that each of these factors imposes on the partition pairs $\{Y_1, Y_2\}$ and $\{W_1, W_2\}$.

5.2. Notation. We set

$$(52) \quad a_1 = -a_2 = a = - \left(\frac{m_a + 1}{2} \right) \alpha_+ - \left(\frac{n_a + 1}{2} \right) \alpha_-,$$

$$m_a \in \{0, 1, \dots, p-2\}, n_a \in \{0, 1, \dots, p'-2\},$$

$$(53) \quad b_1 = -b_2 = b = - \left(\frac{m_b + 1}{2} \right) \alpha_+ - \left(\frac{n_b + 1}{2} \right) \alpha_-,$$

$$m_b \in \{0, 1, \dots, p-2\}, n_b \in \{0, 1, \dots, p'-2\},$$

$$(54) \quad \mu = -\frac{1}{2} m_\mu \alpha_+ - \frac{1}{2} n_\mu \alpha_-, \quad m_\mu \in \{0, 1, \dots, p-2\}, n_\mu \in \{0, 1, \dots, p'-2\},$$

and use the subscript a (b) to indicate the parameters that appear in the conditions on the partition pairs \vec{Y} (\vec{W}) that label the states in the incoming (outgoing) channel that flows towards (away from) the vertex operator insertion. It is useful to note that, in this notation,

$$(55) \quad r_a = m_a + 1, \quad s_a = n_a + 1, \quad r_b = m_b + 1, \quad s_b = n_b + 1$$

Further, to simplify the presentation, we use the notation

$$(56) \quad M_{[\pm, \pm, \pm]} = \frac{1}{2} \left(\pm m_a \pm m_b \pm m_\mu \right), \quad N_{[\pm, \pm, \pm]} = \frac{1}{2} \left(\pm n_a \pm n_b \pm n_\mu \right)$$

5.3. The fusion rules. In the notation of subsection 5.2, the fusion rules are

$$(57) \quad m_a + m_b + m_\mu = 0 \mod 2, \quad n_a + n_b + n_\mu = 0 \mod 2$$

and the triple $\{m_a, m_b, m_\mu\}$ satisfies the triangular conditions

$$(58) \quad m_a + m_b \geq m_\mu, \quad m_b + m_\mu \geq m_a, \quad m_\mu + m_a \geq m_b$$

with analogous triangular conditions for $\{n_a, n_b, n_\mu\}$.

5.4. Bounds on $M_{[\pm, \pm, \pm]}$ and $N_{[\pm, \pm, \pm]}$. For the purposes of the proofs in subsections 5.7 and 5.8, we need to show that $M_{[\pm, \pm, \pm]}$ and $N_{[\pm, \pm, \pm]}$ satisfy the bounds

$$(59) \quad 0 \leq M_{[\pm, \pm, \pm]} \leq p-2, \quad 0 \leq N_{[\pm, \pm, \pm]} \leq p'-2,$$

The lower bounds follow from the lower bounds in the definitions (52–54). The upper bounds are obtained as follows. There are two ways to choose the charge content of the highest weight state of a Virasoro irrep. The first choice is $\alpha_{r,s}$, where

$$\begin{aligned}
(60) \quad 2\alpha_{r,s} &= - \left(r-1 \right) \alpha_+ - \left(s-1 \right) \alpha_-, \\
&\quad r = m+1, \quad s = n+1, \\
&\quad 0 \leq m \leq p-2, \quad 0 \leq n \leq p'-2,
\end{aligned}$$

while the second choice is

$$\begin{aligned}
(61) \quad 2\alpha_{r',s'} &= - \left(r'-1 \right) \alpha_+ - \left(s'-1 \right) \alpha_-, \\
&\quad r' = m'+1, \quad s' = n'+1, \\
&\quad 0 \leq m' \leq p-2, \quad 0 \leq n' \leq p'-2,
\end{aligned}$$

and the two choice are related by

$$(62) \quad r' = p - r, \quad s' = p' - s$$

While the two representations are the same, for the purposes of the proofs in the sequel, we need to use one or the other, as follows.

Scanning a linear conformal block $\mathcal{B}_n^{p,p',\mathcal{H}}$ from left to right, one considers the building block Z_{bb}^ι , $\iota = 1, 2, \dots$, with the Virasoro irrep labeled by $\{r_{\iota-1}, s_{\iota-1}\}$ flowing in from the left, the vertex operator \mathcal{O}_ι of the primary field labeled by $\{r_\mu, s_\mu\}$ in the middle, and the Virasoro irrep labeled by $\{r_\iota, s_\iota\}$ flowing out to the right. Suppose that the charge content of the incoming primary field in $\chi_{\iota-1}$ is fixed¹⁴. The charge content of primary state μ of the vertex operator \mathcal{O}_μ in the middle, and that of the outgoing primary field in χ_ι are not fixed yet, and each can be chosen in one of two equivalent ways. We wish to show that we can choose these charge contents in such a way that the upper bounds in (59) are satisfied. This will simplify our proofs in the sequel.

If $\{m_\mu, n_\mu\}$ and $\{m'_\mu, n'_\mu\}$ are such that the upper bounds in (59) are satisfied, then use this choice. If $\{m_\mu, n_\mu\}$ and $\{m'_\mu, n'_\mu\}$ are such that the upper bounds in (59) are *not* satisfied, we choose the dual representation of the vertex operator in the middle and the outgoing Virasoro irrep¹⁵. In other words, $p - \frac{1}{2} \left(m_{\iota-1} + m_\mu + m_\iota \right) - 2$, that does not satisfy the upper bound, becomes

$$\begin{aligned}
(63) \quad p - \frac{1}{2} \left(m_{\iota-1} + m'_\mu + m'_\iota \right) - 2 &= \\
&= p - \frac{1}{2} \left(m_{\iota-1} + p - m_\mu - 2 + p - m_\iota - 2 \right) - 2 = \\
&= \frac{1}{2} \left(m_{\iota-1} + m_\mu + m_\iota \right) \geq 0
\end{aligned}$$

using the triangular conditions (58), and similarly

$$(64) \quad p' - \frac{1}{2} \left(n_{\iota-1} + n'_\mu + n'_\iota \right) - 2 \geq 0$$

Now the charge content of the outgoing primary field is fixed and goes on to become the incoming primary field of $Z_{bb}^{\iota+1}$ or the primary state of \mathcal{O}_{n+3} . Thus we can always choose the charge contents such that the upper bounds in equations (63) and (64).

¹⁴ Starting from Z_{bb}^1 , we can choose the charge of the highest weight state in \mathcal{O}_0 either way, but for the purposes of this proof, it is sufficient to consider an arbitrary Z_{bb}^ι , $\iota = 1, 2, \dots, n+1$, and take the charge of the primary field in $\chi_{\iota-1}$ to be fixed.

¹⁵ Remember that the charge content of the incoming primary field is given and cannot be changed.

In the following subsections, we consider the conditions that products in the numerator must satisfy to be non-zero ¹⁶.

5.5. $\{\mathbf{Y}_1, \mathbf{W}_1\}_{\text{num}}$. This product vanishes if any factor satisfies

$$(65) \quad E[a - b, Y_1, W_1, \square] - \mu = -M_{[+,-,-]} \alpha_+ - N_{[+,-,-]} \alpha_- + A_{Y_1}^+ \alpha_+ - L_{W_1} \alpha_- = 0,$$

which leads to the zero-conditions

$$(66) \quad A_{\square, Y_1} = -M_{[-,+,+]} + c p - 1, \quad -L_{\square, W_1} = -N_{[-,+,+]} + c p'$$

From the triangular conditions (58), the maximal value of $M_{[-,+,+]}$ is $p - 2$, and the maximal value of $N_{[-,+,+]}$ is $p' - 2$, thus the stronger condition corresponds to $c = 1$, and we obtain two zero-conditions that we can write as one non-zero-condition,

$$(67) \quad W_{1,c}^\top - Y_{1,c+[p-M_{[-,+,+]}-1]}^\top \geq -\left(p' - N_{[-,+,+]} - 1\right)$$

5.6. $\{\mathbf{Y}_1, \mathbf{W}_1\}'_{\text{num}}$. This product vanishes if any factor satisfies

$$(68) \quad E[-a + b, W_1, Y_1, \square] + \mu - 2\alpha_0 = -\frac{1}{2} \left(-m_a + m_b + m_\mu\right) \alpha_+ - \frac{1}{2} \left(-n_a + n_b + n_\mu\right) \alpha_- + A_{W_1} \alpha_+ - L_{Y_1}^+ \alpha_- = 0,$$

which leads to the zero-conditions

$$(69) \quad A_{\square, W_1} = M_{[-,+,+]} + c p, \quad -L_{\square, Y_1} = N_{[-,+,+]} + 1 + c p'$$

Following subsection 5.5, we should set $c = 0$, and (69) translates to the non-zero-condition

$$(70) \quad Y_{1,c}^\top - W_{1,c+M_{[-,+,+]}}^\top \geq -N_{[-,+,+]}$$

5.7. **The remaining six products.** The analysis of the remaining six products is identical to that in subsections 5.5 and 5.6, and it suffices to list the non-zero-condition in each case.

5.7.1. $\{\mathbf{Y}_2, \mathbf{W}_2\}_{\text{num}}$.

$$(71) \quad W_{2,c}^\top - Y_{2,c+[p-M_{[+,-,+]}-1]}^\top \geq -\left(p' - N_{[+,-,+]} - 1\right)$$

5.7.2. $\{\mathbf{Y}_2, \mathbf{W}_2\}'_{\text{num}}$.

$$(72) \quad Y_{2,c}^\top - W_{2,c+M_{[+,-,+]} }^\top \geq -N_{[+,-,+]}$$

5.7.3. $\{\mathbf{Y}_1, \mathbf{W}_2\}_{\text{num}}$.

$$(73) \quad W_{2,c}^\top - Y_{1,c+M_{[+,+,-]}}^\top \geq -N_{[+,+,-]}$$

5.7.4. $\{\mathbf{Y}_1, \mathbf{W}_2\}'_{\text{num}}$.

$$(74) \quad Y_{1,c}^\top - W_{2,c+[p-M_{[+,+,-]}-1]}^\top \geq -\left(p' - N_{[+,+,-]} - 1\right)$$

¹⁶ In writing equations (67), (70), and (71–75), we choose to group terms together in such a way to make the analogy with conditions (37–51), that involve $\{Y_1, Y_2\}$ only, relatively more clear. Basically, M and N in the former are analogues of $(r - 1)$ and $(s - 1)$ in the latter.

5.7.5. $\{\mathbf{Y}_2, \mathbf{W}_1\}_{\text{num}}$.

$$(75) \quad W_{1,c}^\top - Y_{2,c+[p-M_{[+,+,+]}-1]-1}^\top \geq 1 - \left(p' - N_{[+,+,+]} - 1 \right)$$

5.7.6. $\{\mathbf{Y}_2, \mathbf{W}_1\}'_{\text{num}}$.

$$(76) \quad Y_{2,c}^\top - W_{1,c+M_{[+,+,+]}+1}^\top \geq - \left(N_{[+,+,+]} + 1 \right)$$

5.7.7. *Remark.* Equations (74) and (75) make sense as Burge-type conditions because of the bounds in (63) and (64).

5.8. If the denominator is zero, then the numerator is zero. The non-zero-conditions on the $\{Y_i, W_j\}_{\text{num}}$ and $\{Y_i, W_j\}'_{\text{num}}$ products that appear in the numerator can be combined in pairs to produce non-zero-conditions on $\{Y_i, Y_j\}$ and $\{W_i, W_j\}$ pairs, $i \neq j$ pairs also in the numerator, that can be compared to the first and second non-zero-conditions (37) and (44) obtained from the denominator.

5.8.1. $\{\mathbf{Y}_1, \mathbf{W}_1\}_{\text{num}}$ and $\{\mathbf{Y}_2, \mathbf{W}_1\}'_{\text{num}}$. Consider conditions (67) and (76). We eliminate W_1^\top to obtain a non-zero condition on Y_1 and Y_2 by re-writing (67) as

$$(77) \quad W_{1,c+M_{[+,+,+]}+1}^\top - Y_{1,c+[p-M_{[-,+,+]}-1]+M_{[+,+,+]}+1}^\top \geq -p' + N_{[-,+,+]} + 1$$

which combined with condition (76) gives

$$(78) \quad Y_{2,c}^\top - Y_{1,c+[r_a-1]+p}^\top \geq 1 - s_a - p'$$

which is a weak version of condition (37).

5.8.2. $\{\mathbf{Y}_1, \mathbf{W}_2\}_{\text{num}}$ and $\{\mathbf{Y}_2, \mathbf{W}_2\}'_{\text{num}}$. From (73) and (72),

$$(79) \quad Y_{2,c}^\top - Y_{1,c+[r_a-1]}^\top \geq 1 - s_a$$

which is condition (37).

5.8.3. $\{\mathbf{Y}_2, \mathbf{W}_1\}_{\text{num}}$ and $\{\mathbf{Y}_1, \mathbf{W}_1\}'_{\text{num}}$. From (75) and (70),

$$(80) \quad Y_{1,c}^\top - Y_{2,c+[p-r_a]-1}^\top \geq 1 - [p' - s_a],$$

which is condition (44).

5.8.4. $\{\mathbf{Y}_2, \mathbf{W}_2\}_{\text{num}}$ and $\{\mathbf{Y}_1, \mathbf{W}_2\}'_{\text{num}}$. From (71) and (74),

$$(81) \quad Y_{1,c}^\top - Y_{2,c+[p-r_a-1]+p}^\top \geq 1 - [p' - s_a] - p',$$

which is a weak version of condition (44).

5.8.5. $\{\mathbf{Y}_1, \mathbf{W}_1\}'_{\text{num}}$ and $\{\mathbf{Y}_1, \mathbf{W}_2\}_{\text{num}}$. From (70) and (73),

$$(82) \quad W_{2,c}^\top - W_{1,c+r_b-1}^\top \geq 1 - s_b,$$

which is condition (37).

5.8.6. $\{\mathbf{Y}_2, \mathbf{W}_1\}'_{\text{num}}$ and $\{\mathbf{Y}_2, \mathbf{W}_2\}_{\text{num}}$. From (76) and (71),

$$(83) \quad W_{2,c}^\top - W_{1,c+[r_b-1]+p}^\top \geq 1 - s_b - p'$$

which is a weak version of condition (37).

5.8.7. $\{\mathbf{Y}_1, \mathbf{W}_2\}'_{\text{num}}$ and $\{\mathbf{Y}_1, \mathbf{W}_1\}_{\text{num}}$. From (74) and (67),

$$(84) \quad W_{1,c}^T - W_{2,c+[p-r_b-1]+p}^T \geq 1 - [p' - s_b] - p'$$

which is a weak version of condition (44).

5.8.8. $\{\mathbf{Y}_2, \mathbf{W}_2\}'_{\text{num}}$ and $\{\mathbf{Y}_2, \mathbf{W}_1\}_{\text{num}}$. From (72) and (75),

$$(85) \quad W_{1,c}^T - W_{2,c+[p-r_b-1]}^T \geq 1 - [p' - s_b]$$

which is condition (44).

The stronger condition in each of the above cases is one of the Burge conditions. Thus, when the denominator z_{den} of the building block partition function Z_{bb} is non-zero, then the numerator z_{num} is also non-zero. The reverse is not true.

Note that the above result is similar but different from that in [7], where Zamolodchikov argues that **1.** The conformal block \mathcal{B}_n^{gen} is a meromorphic function in Δ_a , the conformal dimension of the Virasoro irrep that flows in a channel, and that \mathcal{B}_n^{gen} has only simple poles at $\Delta_a = \Delta_{a_{r,s}}$, where $a_{r,s} = -\frac{1}{2}[r\alpha_+ + s\alpha_-]$, and **2.** If the fusion rules are satisfied, then the residue at each pole vanishes.

Our result is that **1.** When a summand in Z_{bb} has a zero in the denominator, and the fusion rules are satisfied, then it also has a zero in the numerator. This is independent of Zamolodchikov's statement, since in the latter, the whole sum vanishes rather than just the summand with the zero in the denominator. **2.** Zamolodchikov has argued that \mathcal{B}_n^{gen} has only simple poles, while, as far as we can tell, summands in Z_{bb} can have poles of order greater than 1.

6. AN ISING CONFORMAL BLOCK

In this section, we set $p = 3$ and $p' = 4$, so that the minimal model component $\mathcal{M}^{p,p'}$ of the conformal field theory $\mathcal{M}^{p,p',\mathcal{H}}$ under consideration, is the Ising model. In this case, there are three primary fields to form conformal blocks from. They can be labeled as follows. $\{r, s\} = \{1, 1\}$ is the identity operator $\mathbb{1}$, $\{r, s\} = \{1, 2\}$ is the spin operator σ and $\{r, s\} = \{1, 3\}$ is the thermal operator ψ . Explicit expressions for conformal blocks can be found in [15] and references therein. Consider the 6-point conformal block of σ fields in Figure 6.

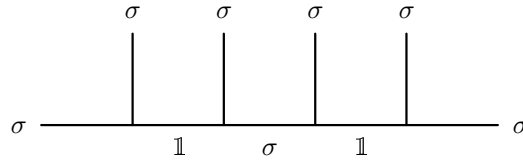


Figure 5. The comb diagram representation of the Ising 6-point conformal block discussed in 6. All external lines correspond to vertex operator insertion of the spin operator σ . The internal channels carry the Virasoro irrep's that correspond to the identity operator, the spin operator, then the identity operator.

In this case, $\alpha_+ = \sqrt{4/3}$, $\alpha_- = -\sqrt{3/4}$, and $\alpha_{1,2} = -\frac{1}{2}\alpha_- = -\frac{1}{2}\sqrt{3/4}$, and following [15],

$$(86) \quad \langle \sigma(z_0) \sigma(z_1) \cdots \sigma(z_5) \rangle =$$

$$\frac{1}{2} \prod_{i=1}^3 \left(z_{2i-2} - z_{2i-1} \right)^{-\frac{1}{8}} \left[\sum_{\substack{t_1=1, \\ t_2, t_3=-1, 1}} \prod_{i=1}^3 t_i \prod_{1 \leq i < j \leq 3} \left(1 - x_{i,j} \right)^{\frac{t_i t_j}{4}} \right]^{\frac{1}{2}},$$

where

$$(87) \quad x_{i,j} = \frac{(z_{2i-2} - z_{2i-1})(z_{2j-2} - z_{2j-1})}{(z_{2i-2} - z_{2j-1})(z_{2j-2} - z_{2i-1})}$$

setting the coordinates $z_0 = 0$, $z_1 = q_1 q_2 q_3$, $z_2 = q_2 q_3$, $z_3 = q_3$, $z_4 = 1$, $z_5 = \infty$,

$$(88) \quad x_{1,2} = \frac{-q_1(1-q_2)}{1-q_1}, \quad x_{1,3} = q_1 q_2 q_3, \quad x_{2,3} = \frac{q_3(1-q_2)}{1-q_2 q_3}.$$

The instanton partition function should equal the product of the Ising conformal block and a contribution from the Heisenberg algebra \mathcal{H} ¹⁷. Using *e.g.* [5], equation (1.9),

$$(89) \quad Z = \left(\prod_{1 \leq i \leq j \leq 3} (1 - q_i \cdots q_j)^{\frac{1}{8}} \right) \langle \sigma(z_0) \sigma(z_1) \cdots \sigma(z_5) \rangle$$

Therefore

$$(90) \quad Z = 1 - \frac{1}{8}q_1 - \frac{1}{8}q_3 - \frac{5}{128}q_1^2 - \frac{1}{8}q_1 q_2 + \frac{1}{64}q_1 q_3 \\ - \frac{5}{128}q_2^2 - \frac{1}{8}q_2 q_3 - \frac{5}{128}q_3^2 + \cdots - \frac{453}{8192}q_1^2 q_2^2 q_3^2 + \cdots$$

Calculating the expansion of Z up to degree 2 in each variable, we find that result coincides with the sum of non-zero terms in the instanton partition function. Using the notation

$$(91) \quad Z_{Nek} \left(Y_1, Y_2 \mid Y_3, Y_4 \mid Y_5, Y_6 \right) = \\ Z_{bb} \left(\emptyset, \emptyset \mid Y_1, Y_2 \right) \cdot Z_{bb} \left(Y_1, Y_2 \mid Y_3, Y_4 \right) \cdot Z_{bb} \left(Y_3, Y_4 \mid \emptyset, \emptyset \right)$$

the $q_1^2 q_2^2 q_3^2$ -term, as an example, is

$$(92) \quad Z_{Nek} \left(2, \emptyset \mid \emptyset, 2 \mid 2, \emptyset \right) + Z_{Nek} \left(2, \emptyset \mid 1, 1 \mid 2, \emptyset \right) + Z_{Nek} \left(2, \emptyset \mid 2, \emptyset \mid 2, \emptyset \right) \\ + Z_{Nek} \left(2, \emptyset \mid 2, \emptyset \mid 1 + 1, \emptyset \right) + Z_{Nek} \left(2, \emptyset \mid 1, 1 \mid 1 + 1, \emptyset \right) = -\frac{453}{8192}$$

while all other terms, that satisfy Proposition 4.1 and the condition $|Y_1| + |Y_2| = |Y_3| + |Y_4| = |Y_5| + |Y_6| = 2$, vanish.

¹⁷ The contribution of the Heisenberg algebra \mathcal{H} is often referred to as *the $U(1)$ factor*.

7. AN EXPLANATION, BASED ON A CONJECTURE, OF WHY WE OBTAIN $\mathcal{M}^{p,p',\mathcal{H}}$
CONFORMAL BLOCKS

As mentioned in section 1, there is a proof of AGT in the context of conformal blocks in $\mathcal{M}^{gen,\mathcal{H}}$ with non-degenerate intermediate Virasoro representations in [5]. In this subsection, we use **1.** results from [5], **2.** Proposition 4.1 of section 4, **3.** that the generating function of Burge pairs is the character of $\mathcal{H}_{r,s}^{p,p',\mathcal{H}}$ [10, 11], and **4.** Conjecture 7.1 below, to explain why restricting the summation to Burge pairs as in (4) leads to conformal blocks in $\mathcal{M}^{p,p',\mathcal{H}}$. Proving **3** would amount to proving that restricting to Burge pairs leads to conformal blocks in $\mathcal{M}^{p,p',\mathcal{H}}$, but this is beyond the scope of this work.

Consider the Verma module \mathcal{H}_a over $\mathcal{V}^{gen} \oplus \mathcal{H}$ generated by highest-weight vector $|a\rangle$,

$$(93) \quad L_k|a\rangle = a_k|a\rangle = 0, \quad k > 0, \quad L_0|a\rangle = \Delta_a|a\rangle,$$

where L_k , and a_k , $k \in \mathbb{Z}$, are generators of \mathcal{V}^{gen} and \mathcal{H} , respectively, and

$$(94) \quad \Delta_a = \frac{1}{4} \left(b_{gen} + \frac{1}{b_{gen}} \right)^2 - a^2, \quad b_{gen} = \left(\frac{\epsilon_2}{\epsilon_1} \right)^{\frac{1}{2}}.$$

Conformal blocks are defined in terms of vertex operators $\mathcal{O}_\mu(z) : \mathcal{H}_a \mapsto \mathcal{H}_b$, that in turn are defined by the commutation relations

$$(95) \quad [L_k, \mathcal{O}_\mu(z)] = z^{k+1} \partial_z \mathcal{O}_\mu(z) + i(k+1) \Delta_{\mu'} q^k \mathcal{O}_\mu(q), \quad \mu' = \mu - \frac{\epsilon_1 + \epsilon_2}{2}$$

as well as

$$(96) \quad [a_k, \mathcal{O}_\mu(z)] = i\mu z^k \mathcal{O}_\mu(z), \quad k < 0, \quad [a_k, \mathcal{O}_\mu(z)] = i(Q - \mu) z^k \mathcal{O}_\mu(z), \quad k > 0$$

AGT was proven in [5] for generic central charge c_{gen} , in the following sense

Proposition 7.1. *Following [5], there exists an orthogonal basis $J_{\vec{Y}} \in \mathcal{H}_a$ labeled by pairs of Young diagrams such that the matrix elements of vertex operator \mathcal{O}_μ satisfy*

$$(97) \quad \frac{\langle J_{\vec{Y}} | \mathcal{O}_\mu | J_{\vec{W}} \rangle}{\langle J_{\vec{B}} | \mathcal{O}_\mu | J_{\vec{B}} \rangle} = z_{num} \left(\vec{a}, \vec{Y} \mid \mu \mid \vec{b}, \vec{W} \right)$$

where $\vec{a} = \{a, -a\}$, $\vec{b} = \{b, -b\}$.

From this proposition, it follows that

$$(98) \quad \langle J_{\vec{Y}} | J_{\vec{W}} \rangle = z_{norm} \left(\vec{a}, \vec{Y} \right) \delta_{\vec{Y}, \vec{W}}$$

The vectors $J_{\vec{Y}}$ can be written in the standard basis of the Verma module,

$$(99) \quad J_{\vec{Y}} = \sum_{\lambda, \mu} C_{\vec{Y}}^{\lambda, \mu} L_{-\lambda_1} L_{-\lambda_2} \cdots a_{-\mu_1} a_{-\mu_2} \cdots |a\rangle,$$

where the summation is over partition pairs $\{\lambda, \mu\}$ such that $|\lambda| + |\mu| = |Y_1| + |Y_2|$. The coefficients $C_{\vec{Y}}^{\lambda, \mu}$ depend on the parameters a, b_{gen} . In [5, Corollary 3.8], it was proven that $C_{\vec{Y}}^{\lambda, \mu}$ is a polynomial in a . In this section, we need the following conjecture¹⁸

Conjecture 7.1. *The coefficients $C_{\vec{Y}}^{\lambda, \mu}$ are Laurent polynomials in b_{gen} .*

Conjecture 7.1 is motivated by the explicit examples of the vectors $J_{\vec{Y}}$ [5]. Further motivation is provided by the relation between Jack symmetric functions J_Y^α [16] and $J_{\vec{Y}}$, for $\alpha = b_{gen}^2$ [5]. Namely, from Macdonald's conjectures, proved by Haiman [17], the coefficients of J_Y^α , in the standard basis, are polynomial in α , so it is natural to expect that $J_{\vec{Y}}$ satisfies an analogous property.

¹⁸ This conjecture is not original to this work. It is standard in the community, although not written in the literature.

Assuming Conjecture 7.1, we can set $b_{gen} \rightarrow b_{p,p'}$ and $a \rightarrow a_{r,s}$, as defined in subsection 3.2. Thus, we can consider $J_{\vec{Y}}$ as vectors in the module over $\mathcal{H} \oplus \mathcal{V}^{p,p'}$. In this case, the Verma module \mathcal{H}_a becomes reducible, the maximal submodule Ker_a is a kernel of the Shapovalov form $\langle \cdot | \cdot \rangle$ on \mathcal{H}_a , and the irreducible quotient is $\mathcal{H}_{r,s}^{p,p',\mathcal{H}} = \mathcal{H}_{r,s}^{p,p'} \otimes \mathcal{F}$.

It follows from (4) and (98) that \vec{Y} satisfy condition (24) if and only if the vector $J_{\vec{Y}}$ is not in the kernel Ker_a . Then to each Burge pair \vec{Y}_{Burge} one can consider $J_{\vec{Y}}$ as an element of quotient $\mathcal{H}_{r,s}^{p,p',\mathcal{H}}$. The vectors $J_{\vec{Y}} \in \mathcal{H}_{r,s}^{p,p',\mathcal{H}}$, for a Burge pair \vec{Y}_{Burge} , are linearly independent since they are orthogonal. It was proven in [10, 11], that the generating function of Burge pairs give the character of $\mathcal{H}_{r,s}^{p,p',\mathcal{H}}$. Therefore the vectors $J_{\vec{Y}_{Burge}}$ form a basis in $\mathcal{H}_{r,s}^{p,p',\mathcal{H}}$. This is the point of this subsection.

Finally we note that the norms and matrix elements of \mathcal{O}_μ depend on the parameters $\{a, b, \mu, \epsilon_1, \epsilon_2\}$ algebraically. Therefore, since (97) was proven for a generic central charge, it holds for the $\mathcal{M}^{p,p',\mathcal{H}}$ models, and the expression that we obtain for $\mathcal{B}_n^{p,p',\mathcal{H}}$, by summing over Burge pairs, holds.

8. GENERIC MODEL CONFORMAL BLOCKS WITH DEGENERATE INTERMEDIATE REPRESENTATIONS

In generic models with a generic central charge c_{gen} , $z_{norm}[\vec{a}, \vec{Y}]$ can have zeros due to degenerate \mathcal{V}^{gen} representations in the intermediate channels. Since $\vec{a} = \{a, -a\}$, setting $2a = 2a_{r,s} = -r\beta_+ - s\beta_-$, we can study these zeros just as in section 4. Since the central charge is generic we have

$$(100) \quad C_+ \beta_+ + C_- \beta_- = 0,$$

if and only if $C_+ = C_- = 0$, and only the $\{\mathbf{Y}_1, \mathbf{Y}_2\}_{den}$ factors can be zero. Proceeding from the two zero-conditions (100), we obtain

Proposition 8.1. $z_{norm}[\vec{a}, \vec{Y}] \neq 0$ if and only if $Y_{2,c}^\top - Y_{1,c+r-1}^\top \geq 1 - s$

The proof of Proposition 8.1 follows the same line of arguments as the proof of Proposition 4.1 in section 4. From Proposition 8.1, we obtain

Proposition 8.2.

$$(101) \quad \mathcal{B}_n^{degen,\mathcal{H}} = \sum'_{\vec{Y}^1, \dots, \vec{Y}^n} \prod_{\iota=1}^{n+1} q_\iota^{|\vec{Y}^\iota|} Z_{bb} \left(\vec{a}^{\iota-1}, \vec{Y}^{\iota-1} \mid \mu^\iota \mid \vec{a}^\iota, \vec{Y}^\iota, \right)$$

where $\mathcal{B}_n^{degen,\mathcal{H}}$ is an n -channel generic model conformal block, such that some of the channels carry degenerate intermediate representations, and \sum' indicates that, for channels that carry degenerate representations, the sum is restricted to partition pairs that satisfy Proposition 8.1.

The proof of Proposition 8.2 is based on the same line of arguments as in section 7 but *without* requiring a conjecture analogous to Conjecture 7.1. Indeed, since the coefficients of $J_{\vec{Y}}$ are polynomial in a , we can set $a = a_{r,s}$ in (99). The vectors $J_{\vec{Y}}$ for which $z_{norm}[\vec{a}, \vec{Y}] = 0$, belong to the kernel of the Shapovalov form on the Verma module \mathcal{H}_a . Let $\mathcal{H}_{r,s}^{gen}$ denote the irreducible quotient of \mathcal{H}_a . The vectors $J_{\vec{Y}}$, where \vec{Y} satisfy Proposition 8.1, project to the module $\mathcal{H}_{r,s}^{gen}$. In [11], Feigin *et al.* proved that the generating function of the \vec{Y} pairs that satisfy Proposition 8.1 is the character of $\mathcal{H}_{r,s}^{gen}$, therefore the corresponding vectors $J_{\vec{Y}}$ form a basis in $\mathcal{H}_{r,s}^{gen}$. Using (97) and (98), we obtain the expression (101) for the conformal block for degenerate representations.

9. COMMENTS AND REMARKS

9.1. q - \mathfrak{gl}_∞ Ding-Iohara. Let \mathcal{E} be the algebra called q -deformed \mathfrak{gl}_∞ in [11], and Ding-Iohara in [18]¹⁹. Following [18], operators in the rank-2 representations $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$ of \mathcal{E} generate the sum of a q -deformed Virasoro algebra and a q -deformed Heisenberg algebra. On the other hand, following [11] [Theorem 3.8], for special values of parameters u_1 and u_2 , as well as q_1 and q_3 of \mathcal{E} , this representation has a sub-representation with a basis labeled by Burge pairs. In is natural to expect that in the limit $q_1, q_3 \rightarrow 1$, the basis constructed in [11] reduces to the basis $J_{\vec{\gamma}}$ described in section 7.

9.2. Higher-rank AGT-W. AGT was extended to theories based on the higher rank algebras $\mathcal{W}_N \oplus \mathcal{H}$, $N > 2$, by Wyllard in [19], and by Mironov and Morozov in [20]. In this note, we chose to simplify the presentation by focusing on Virasoro minimal models, but we expect that our analysis extends without essential modification to minimal models based on \mathcal{W}_N algebras with $N > 2$. We conjecture that the restricted partitions that are relevant to these extended cases are those that appeared in [11].

9.3. The work of Alkalaev and Belavin. In [21], Alkalaev and Belavin independently suggested the Virasoro result in (4) in the 4-point conformal block case. They proved a proposition equivalent to 4.1, made the same comment on conformal blocks in generic models with degenerate intermediate representations as in section 8, albeit without proving an analogue of Proposition 8.2 and made the same W_N conjecture as in subsection 9.2.

9.4. Previous works on AGT in minimal models. There are two previous works on AGT in minimal models that we are aware of. In [8], Santachiara and Tanzini identify Moore-Read wave functions, which are minimal model conformal blocks of $\{r, s\} = \{1, 2\}$ and $\{2, 1\}$ vertex operators, with Nekrasov instanton partition functions, AGT is applied without modification to these conformal blocks and ill-defined expressions are made well-defined using a deformation scheme, as outlined in subsection 1.3. In [22], Estienne, Pasquier, Santachiara and Serban interpret $W_n \oplus \mathcal{H}$ minimal model conformal blocks of $\{r, s\} = \{1, 2\}$ and $\{2, 1\}$ vertex operators as wave functions of a trigonometric Calogero-Sutherland models with non-trivial braiding properties, and find that the excited states are characterized by $(n+1)$ -partitions, just as in AGT. While Estienne *et al.* use different notation from ours, preliminary checks indicate that their partitions can be translated to the Burge pairs used in this note, for $n = 2$, and $\{r, s\} = \{1, 2\}$ or $\{2, 1\}$.

9.5. Geometry. Let $\mathcal{M}(r, N)$ be the moduli space of $U(r)$ instantons on \mathbb{R}^4 . The instanton partition function for $\prod_{i=1}^n U(2)_i$ gauge theory equals the generating function of equivariant integrals over $\mathcal{M}(2, N_1) \times \cdots \times \mathcal{M}(2, N_n)$, where the equivariant integral is taken with respect to the torus $\mathbb{T} = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)_1^2 \times (\mathbb{C}^*)_2^2 \times \cdots \times (\mathbb{C}^*)_n^2$, where the first $(\mathbb{C}^*)^2$ acts on \mathbb{C}^2 , and $(\mathbb{C}^*)_i^2$ acts on the i -th instanton moduli space $\mathcal{M}(2, N_i)$ by constant gauge transformation. These equivariant integrals are computed using localization and are given by the sum over torus fixed points. These points were labeled by n pairs of Young diagrams $\vec{Y}^1, \dots, \vec{Y}^n$. The parameters ϵ_1, ϵ_2 , and \vec{a}^i are the coordinates on $\mathfrak{t} = \text{Lie}(\mathbb{T})$. In the $\mathcal{M}^{p,p',\mathcal{H}}$ case, ϵ_1 and ϵ_2 are linearly-dependent, and a_j^i is given by (18). Geometrically, this means that we are considering the one-dimensional subgroup $\mathbb{C}_{\epsilon_1, \epsilon_2, \vec{a}^i}^* \subset \mathbb{T}$.

The function $z_{\text{norm}}[\vec{a}, \vec{Y}]$ is the determinant of the vector field with coordinates $\{\epsilon_1, \epsilon_2, a_1, a_2\}$ on the tangent space of the point labeled by \vec{Y} . The condition $z_{\text{norm}}[\vec{a}, \vec{Y}] \neq 0$ is equivalent to the fact that corresponding point is an isolated fixed point of the one dimensional torus $\mathbb{C}_{\epsilon_1, \epsilon_2, \vec{a}^i}^*$. Therefore, summing over Burge pairs is equivalent to summing over the *isolated* torus fixed points.

¹⁹ Also called ‘quantum toroidal $\mathfrak{gl}(1)$ algebra’, ‘elliptic Hall algebra’, *etc.* in the literature.

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